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FIXED POINT THEOREMS FOR COMPATIBLE MAPPINGS ON PARTIALLY ORDERED METRIC SPACES

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Abstract

In this paper, the results are then employe to prove existence and uniqueness theorems for common fixed points of compatible mappings on partially ordered metric spaces. The paper begins with basic definitions concerning compatible, weakly compatible, and partially weakly compatible mappings, as well as monotone mappings on ordered metric spaces. Several auxiliary lemmas are established to analyze convergence behavior and asymptotic commutativity properties.

1 Introduction

Fixed point theory constitutes a fundamental area of nonlinear analysis and has been extensively developed in metric spaces and their generalizations. The classical Banach contraction principle guarantees the existence and uniqueness of a fixed point for a contractive self-mapping on a complete metric space [1]. Since then, numerous extensions have been obtained by [5, 6].

Results of this type show that monotone non-decreasing mappings satisfying suitable contractive conditions often possess fixed points when the space is complete [2, 7]. The order relation plays a crucial role by restricting the contractive condition to comparable elements, thereby enlarging the class of admissible mappings.

2 Preliminaries and Definitions

In this section, we present the fundamental definitions and basic concepts required for the development of the main results of this paper. These notions establish the analytical framework concerning compatibility, order structure, and monotonicity of mappings in partially ordered metric spaces, which will be used throughout the subsequent theorems and proofs.

2.1 Definition (Compatible Mappings) [3]

Let (X, d) be a metric space endowed with a partial order. Two mappings T, S : X → X are called compatible if for every sequence $\{x_n\} \subset X$ such that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} S(x_n) = t,$$

$$\lim_{n\to\infty} d(T(S(x_n)), S(T(x_n))) = 0.$$

2.2 Definition (Weakly Compatible Mappings) [4]

Two mappings T, S : X → X are said to be weakly compatible if they commute at their coincidence points; that is,

$$Tx = Sx \Rightarrow T(Sx) = S(Tx).$$

2.3 Definition (Partially Weakly Compatible Mappings) [9]

Two mappings are called partially weakly compatible if they commute at common fixed points; that is,

$$Tx = Sx = x \Rightarrow T(Sx) = S(Tx) = x.$$

2.4 Definition (Increasing and Monotone Mappings) [7]

A mapping T : X → X is called monotone non-decreasing if $x \le y$ implies $T(x) \preccurlyeq T(y)$.

$$x \preceq y \Rightarrow T(x) \preceq T(y).$$

This property ensures that iterates $x, T(x), T^2(x), \ldots$form an ordered sequence, greatly simplifying convergence analysis.

3 Lemmas for Compatible Mappings on Ordered Metric Spaces

Lemma 3.1 (Monotone Convergence Lemma) [7]

Let (X, d) be complete and let T: X → X be monotone non-decreasing. If $\{x_n\}$ is a sequence such that

Let $(X, d, \preceq)$be complete and let $T: X \to X$ be monotone.

If $\{x_n\}$ is a sequence such that

and $x_n \to x^*$, then

and $x_n \to x^*$, then

Proof.

Since the order is closed with respect to limits in a partially ordered metric space, the inequality $x_n \preceq x_{n+1}$ for all n and the convergence $x_{n+1} \to x^*$ imply

$$x_n \preceq x^*.$$

Thus, the entire sequence is bounded above by its limit.

Lemma 3.2 (Compatibility Implies Asymptotic Commutativity) [8]

If T and S are compatible mappings on a partially ordered metric space and $\{x_n\}$ satisfies

$$x_n \to x^*, T(x_n) \to x^*, S(x_n) \to x^*,$$

then

$$T(S(x_n)) \to T(x^*), S(T(x_n)) \to S(x^*).$$

Proof.

$$d(T(S(x_n)), S(T(x_n))) \to 0.$$

Since

$$S(x_n) \to x^*, T(x_n) \to x^*,$$

continuity of T and S at the limit point yields

$$T(S(x_n)) \to T(x^*), S(T(x_n)) \to S(x^*).$$

4 Main Results

Theorem 4.1 (Existence of Common Fixed Point for Compatible Mappings in a Partially Ordered Metric Space)

Let (X, d) be a complete metric space endowed with a partial order. Let T, S: X → X be two compatible mappings satisfying:

a) T and S are monotone non-decreasing;
b) There exists $x_0 \in X$ with
c) $x_0 \preceq T(x_0), x_0 \preceq S(x_0)$;
d) There exists $0 < k < 1$ such that for all comparable x, y ∈ X,
e) $d(Tx, Sy) \leq k \max\{d(x,y), d(x,Tx),\ d(y,Tx), \frac{d(x,Sy)+d(y,Sy)}{2}\}$

Then T and S have a common fixed point in X.

Proof.

Start with the initial point $x_0 \in X$ satisfying

$$x_0 \preceq T(x_0), x_0 \preceq S(x_0).$$

Define the iterative sequence

$$x_1 = T(x_0), x_2 = S(x_1), x_3 = T(x_2), x_4 = S(x_3), \ldots$$

Since both mappings are monotone non-decreasing and

$$x_0 \preceq x_1,$$

we have

$$x_1 = T(x_0) \preceq T(x_1) = x_2', x_2 = S(x_1) \preceq S(x_1) = x_2.$$

Inductively, the entire sequence is monotone increasing:

$$x_0 \preceq x_1 \preceq x_2 \preceq x_3 \preceq \cdots .$$

Let

$$d_n = d(x_n, x_{n+1}).$$

Using the contractive inequality for comparable elements,

$$d_{n+1} = d(x_{n+1}, x_{n+2})$$
$$= d\big(T(x_n), S(x_{n+1})\big)$$
$$\preceq k\, max\{d(x_n, x_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_n), \frac{d(x_n, Sx_{n+1}) + d(x_{n+1}, Sx_{n+1})}{2}\}$$
$$\preceq k \max\{d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+1}), \frac{d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+2})}{2}\}$$
$$\preceq k\, d(x_n, x_{n+1}) = k\, d_n.$$

Thus $\{d_n\}$ is a strictly decreasing geometric sequence:

$$d_n \preceq k^n d_0.$$

Hence $d_n \to 0$.

To show that $\{x_n\}$ is Cauchy, let $m > n$.

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$$d(x_n, x_m) \le d_n + d_{n+1} + \cdots + d_{m-1} \le d_0(k^n + k^{n+1} + \cdots) = \frac{d_0 k^n}{1-k}.$$

Since $k^n \to 0$, the sequence is Cauchy. Completeness of X gives

$$x_n \to x^*.$$

Now we show

$$T(x^*) = x^* = S(x^*).$$

Since

$$x_{2n} = T(x_{2n-1}) \to x^*, x_{2n+1} = S(x_{2n}) \to x^*,$$

we have

$$T(x_n) \to x^*, S(x_n) \to x^*.$$

Compatibility implies

$$d(T(S(x_n)), S(T(x_n))) \to 0.$$

Taking limits,

$$T(x^*) = S(x^*).$$

Since

$$x^* = \lim\ x_n = \lim\ T(x_{n-2}) = \lim\ S(x_{n-1}),$$

and continuity at the limit gives

$$T(x^*) = x^*, S(x^*) = x^*.$$

Thus x* is a common fixed point.

Theorem 4.2 (Uniqueness of Common Fixed Point for Compatible Mappings on Partially Ordered Metric Spaces)

Let (X, d) be a complete metric space endowed with a partial order. Let T, S : X → X be compatible and monotone non-decreasing mappings such that

1. there exists $x_0 \in X$ with

$$x_0 \preceq T(x_0), x_0 \preceq S(x_0);$$

2. for some $k \in (0,1)$,

$$d(Tx, Sy) \leq k\, d(x, y) \text{for comparable } x, y \in X;$$

then, if a common fixed point exists, it is unique.

Proof.

Assume that a common fixed point exists. Let $x^* \in X$ be such that

$$T(x^*) = x^* = S(x^*).$$

Suppose, for the sake of contradiction, that there exists another common fixed point $y^* \in X$ with $y^* \neq x^*$ and

$$T(y^*) = y^* = S(y^*).$$

Since the space is partially ordered, consider the case when the two points are comparable (as is standard in ordered fixed-point arguments). Without loss of generality, assume

$$x^* \preceq y^*.$$

Because $T$and $S$are monotone non-decreasing, they preserve the order of comparable elements. Hence,

$$x^* \preceq y^* \ \Rightarrow\ T(x^*) \preceq T(y^*), S(x^*) \preceq S(y^*).$$

But both points are fixed points, so this reduces to

Now apply the given contractive condition to the comparable pair (x^*, y^*):

which is consistent.

Using the fixed-point property,

$$d(Tx^*, Sy^*) \leq k\, d(x^*, y^*).$$

we obtain

$$Tx^* = x^*, Sy^* = y^*,$$

Rearranging,

$$d(x^*, y^*) \le k\, d(x^*, y^*).$$

Since k ∈ (0,1), we have 1 − k > 0, therefore the only possibility is

$$(1 - k)\, d(x^*, y^*) \le 0.$$

Because d is a metric, this implies

$$d(x^*, y^*) = 0.$$

This contradicts the assumption that the two fixed points are distinct. Hence no two different common fixed points can exist.

Therefore, whenever a common fixed point exists, it must be unique.

Corollary 4.2.1

Under the same assumptions as Theorem 4.2, if T or S is continuous, then the common fixed point is unique even without assuming explicit compatibility at every point.

Reason.

Continuity together with weak compatibility implies compatibility at coincidence points, which is sufficient.

Corollary 4.2.2

If, in addition,

$$d(Tx, Ty) \le k\, d(x, y) \text{ and } d(Sx, Sy) \le k\, d(x, y),$$

then the unique fixed point of either mapping is a unique common fixed point of both.

4.3 Discussion and Observations

1. Compatibility is stronger than weak compatibility; it imposes limit-commutativity, not merely commutativity at coincidence points.
2. In a partially ordered metric space, the order relation significantly reduces the assumptions required for fixed-point existence.
3. The alternating sequence construction

$$x_1 = T(x_0), x_2 = S(x_1), x_3 = T(x_2), \dots$$

is particularly effective when both mappings are monotone non-decreasing.

4. The contractive condition

$$d(Tx, Sy) \leq kd(x, y)$$

is sufficient to collapse the distance between iterates at a geometric rate.

5. The completeness of the space ensures convergence of the iterative sequence constructed from alternate applications of T and S.
6. Uniqueness of the fixed point follows from the inequality

$$d(x^*, y^*) \leq kd(x^*, y^*)$$

which forces the distance between any two fixed points to be zero.

4.4 Conclusion

In this paper, we established several existence and uniqueness results for common fixed points of compatible and weakly compatible self-mappings on partially ordered metric spaces. The obtained theorems extend classical contraction principles by incorporating order structures, monotonicity, and compatibility conditions between mappings. By constructing monotone iterative sequences generated through alternate applications of the mappings and employing contractive conditions defined on comparable elements, we proved the convergence of the sequences to a common fixed point under relatively weaker assumptions than those required in classical metric space settings.

Furthermore, uniqueness of the common fixed point was obtained through appropriate contractive inequalities, demonstrating that the distance between any two fixed points must vanish. The auxiliary lemmas developed in this paper ensured stability of limits, preservation of order, and asymptotic commutativity of compatible mappings, thereby providing a rigorous analytical framework for the main results.

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